Transversality will be fundamental to the topological theory of subsequent chapters, so be sure to devote some time now toward developing your intuition. The best procedure is probably just to draw several dozen pictures of intersecting manifolds in 2- and 3-space. You will find that transversal intersections are rather prosaic but very dependable and clean. But see how utterly bizarre nontransversal intersections can be—and examine your constructions in order to understand precisely where transversality is violated. We have started you off with a series of tame, but typical, examples.

EXERCISES

1. (a) Suppose that $A\colon \mathbf{R}^k \longrightarrow \mathbf{R}^n$ is a linear map and V is a vector subspace of $\mathbf{R}^n$. Check that $A \pitchfork V$ means just $A(\mathbf{R}^k) + V = \mathbf{R}^n$.
   (b) If V and W are linear subspaces of $\mathbf{R}^n$, then $V \pitchfork W$ means just $V + W = \mathbf{R}^n$.

2. Which of the following linear spaces intersect transversally?
   (a) The xy plane and the z axis in $\mathbf{R}^3$.
   (b) The xy plane and the plane spanned by $\{(3, 2, 0), (0, 4, -1)\}$ in $\mathbf{R}^3$.
   (c) The plane spanned by $\{(1, 0, 0), (2, 1, 0)\}$ and the y axis in $\mathbf{R}^3$.
   (d) $\mathbf{R}^k \times \{0\}$ and $\{0\} \times \mathbf{R}^l$ in $\mathbf{R}^n$. (Depends on k, l, n.)
   (e) $\mathbf{R}^k \times \{0\}$ and $\mathbf{R}^l \times \{0\}$ in $\mathbf{R}^n$. (Depends on k, l, n.)
   (f) $V \times \{0\}$ and the diagonal in $V \times V$.
   (g) The symmetric ($A^t = A$) and skew symmetric ($A^t = -A$) matrices in $M(n)$.

3. Let V_1, V_2, V_3 be linear subspaces of $\mathbf{R}^n$. One says they have "normal intersection" if $V_i \pitchfork (V_j \cap V_k)$ whenever $i \neq j$ and $i \neq k$. Prove that this holds if and only if

$$\operatorname{codim}(V_1 \cap V_2 \cap V_3) = \operatorname{codim} V_1 + \operatorname{codim} V_2 + \operatorname{codim} V_3$$

*4. Let X and Z be transversal submanifolds of Y. Prove that if $y \in X \cap Z$, then

$$T_y(X \cap Z) = T_y(X) \cap T_y(Z).$$

("The tangent space to the intersection is the intersection of the tangent spaces.")

*5. More generally, let $f\colon X \longrightarrow Y$ be a map transversal to a submanifold Z in Y. Then $W = f^{-1}(Z)$ is a submanifold of X. Prove that $T_x(W)$ is the preimage of $T_{f(x)}(Z)$ under the linear map $df_x\colon T_x(X) \longrightarrow T_{f(x)}(Y)$.

("The tangent space to the preimage of Z is the preimage of the tangent space of Z.") (Why does this imply Exercise 4?)

6. Suppose that X and Z do not intersect transversally in Y. May $X \cap Z$ still be a manifold? If so, must its codimension still be codim X + codim Z? (Can it be?) Answer with drawings.

***7.** Let $X \xrightarrow{f} Y \xrightarrow{g} Z$ be a sequence of smooth maps of manifolds, and assume that g is transversal to a submanifold W of Z. Show $f \pitchfork g^{-1}(W)$ if and only if $g \circ f \pitchfork W$.

8. For which values of a does the hyperboloid defined by $x^2 + y^2 - z^2 = 1$ intersect the sphere $x^2 + y^2 + z^2 = a$ transversally? What does the intersection look like for different values of a?

9. Let V be a vector space, and let Δ be the diagonal of $V \times V$. For a linear map $A: V \longrightarrow V$, consider the graph $W = \{(v, Av) : v \in V\}$. Show that $W \pitchfork \Delta$ if and only if $+1$ is not an eigenvalue of A.

10. Let $f: X \longrightarrow X$ be a map with fixed point x; that is, $f(x) = x$. If $+1$ is not an eigenvalue of $df_x : T_x(X) \longrightarrow T_x(X)$, then x is called a *Lefschetz fixed point* of f. f is called a *Lefschetz map* if all its fixed points are Lefschetz. Prove that if X is compact and f is Lefschetz, then f has only finitely many fixed points.

11. A theorem of analysis states that every closed subset of $\mathbf{R}^k$ is the zero set of some smooth function $f: \mathbf{R}^k \longrightarrow \mathbf{R}$. Use this theorem to show that if C is any closed subset of $\mathbf{R}^k$, then there is a submanifold X of R^{k+1} such that $X \cap \mathbf{R}^k = C$. [Here we consider $\mathbf{R}^k$ as a submanifold of $\mathbf{R}^{k+1}$ via the usual inclusion $(a_1, \ldots, a_k) \longrightarrow (a_1, \ldots, a_k, 0)$.] Because closed sets may be extremely bizarre, this shows how bad nontransversal intersections can be.

§6 Homotopy and Stability

A great many properties of a map are not altered if the map is deformed in a smooth manner. Intuitively, one smooth map $f_1 : X \longrightarrow Y$ is a deformation of another $f_0 : X \longrightarrow Y$ if they may be joined by a smoothly evolving family of maps $f_t : X \longrightarrow Y$. (See Figure 1-20.) The precisely formulated definition is one of the fundamental concepts of topology. Let I denote the unit interval [0, 1] in $\mathbf{R}$. We say that f_0 and f_1 are *homotopic*, abbreviated $f_0 \sim f_1$, if there exists a smooth map $F: X \times I \longrightarrow Y$ such that $F(x, 0) = f_0(x)$ and $F(x, 1) = f_1(x)$. F is called a *homotopy* between f_0 and f_1. Homotopy is